"Two straight lines are drawn either from the same fixed point or from two fixed points, in the same direction or in such a way as to form a fixed angle; the lengths of these lines are in a constant ratio to one another or their rectangle is constant. If the extremity of one of them describes a plane locus given in position, the extremity of the second will also describe a plane locus, given in position, which is either of the same or of different species from the first."

A particular proposition included in this general one may be formulated as follows:

Through a point O draw lines $OP_1, OP_2, OP_3, \cdots$ to the various points $P_1, P_2, P_3, \cdots$ on a circle.[1] *Divide $OP_1, OP_2, OP_3, \cdots$ internally at $Q_1, Q_2, Q_3, \cdots$ respectively, and such that $OP_1 : OQ_1 = OP_2 : OQ_2 = \cdots =$ a const., and externally at $R_1, R_2, R_3, \cdots$ respectively, such that $OP_1 : OR_1 = OP_2 : OR_2 = \cdots =$ a const. Then the locus of the Q's is a circle,*[1] *and the locus of the R's is a circle.*[1]

Here O is the external center of similitude of the circles (P) and (Q), and the internal center of similitude of the circles (P) and (R). It is exactly such a proposition which Simson and others (*l. c.*) consider in their restorations in various cases when the circles (1) are exterior to one another; (2) intersect; (3) are such that one is inside of the other. The property of parallel radii joining corresponding points of the pairs of circles, arises in the course of the proofs.

My earlier argument that Apollonius was familiar with the centers of similitude of circles and some of their chief properties has thus been reinforced through consideration of another of his works.

A CIRCLE THEOREM.

By ROGER A. JOHNSON, Adelbert College, Western Reserve University.

THEOREM. *If three equal circles are drawn through a point, the circle through their other three intersections is equal to each of them.*

Proof. Denote the centers of the circles (see figure 1 of the next paper) by C_1, C_2, C_3, the intersections of C_2 and C_3 by O and P_1, those of C_3 and C_1 by O and P_2, those of C_1 and C_2 by O and P_3. Then $OC_2P_1C_3$ is a rhombus, and so is $OC_3P_2C_1$. Hence, C_2P_1 and C_1P_2 are equal and parallel, $C_1C_2P_1P_2$ is a parallelogram, and P_1P_2 is equal to C_1C_2. Thus the triangles $C_1C_2C_3$ and $P_1P_2P_3$ are congruent, and have equal circumcircles. But the circumcircle of the former has its center at O, and is equal to each of the given circles. Hence, the circle through P_1, P_2, P_3 is equal to each of the given circles.

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the ratio of whose distances from two fixed points is constant, is either a straight line or a circle —the Circle of Apollonius. Eutocius gives the proof of Apollonius for this latter locus (Apollonius, ed. Heiberg, Vol. 2, pp. 180–185). The name "Circle of Apollonius" is, however, a misnomer, since the construction of this locus connected with his name appears in exactly the same form at a much earlier date in Aristotle's *Meteorologica*, III, 5, 376 f.

[1] "Circle" is here considered as a curved line. The cases of this proposition when we substitute "straight line" for "circle" were discussed by Euclid in Propositions 35–36 of his *Data* (Simson's edition, Prop. 39—for example: *Elements of Elucid . . . also Euclid's Data*, 9th ed., Edinburgh, 1793, pp. 393–394).

COROLLARY. *Each of the four points O, P_1, P_2, P_3, is the orthocenter of the triangle of the other three, and the set of points has all the well-known properties of an orthocentric system.*

Singularly enough, this remarkable theorem appears to be new. A rather cursory search in several of the treatises on modern elementary geometry fails to disclose it, and the author has not yet found any person to whom it was known. On the other hand, the figure is so simple (especially as it can be drawn and the theorem verified with a coin or other circular object) that it seems almost out of the question that the fact can have escaped detection. Even if geometers have overlooked it, someone must have noticed it in casually drawing circles. But if this were the case, it seems like a theorem of sufficient interest to receive some prominence in the literature, and therefore ought to be well known. It is hoped that if any reader recognizes the theorem, or knows where it has already been given, he will report the same. Of course, the converse theorem that the four circumcircles of an orthocentric system are equal, is well known.

REMARKS ON THE FOREGOING CIRCLE THEOREM.

By ARNOLD EMCH, University of Illinois.

1. The foregoing theorem proved by Mr. Johnson gains additional interest in connection with the theory of circular inversion.

Before this fact is pointed out, another proof of the theorem, equally notable on account of its extreme simplicity, will be given. Using the same notation as Mr. Johnson, and denoting the given circles through O by $\alpha_1, \alpha_2, \alpha_3$, as shown[1] in Fig. 1, we have $\measuredangle OP_3P_2 = \measuredangle OP_1P_2$, because the circles α_1 and α_3 are equal and have the common chord OP_2 subtending those angles. Likewise, $\measuredangle OP_2P_3 = \measuredangle OP_1P_3$. Consequently $\measuredangle OP_3P_2 + \measuredangle OP_2P_3 = \measuredangle OP_1P_2 + \measuredangle OP_1P_3 = \measuredangle P_2P_1P_3$, so that $\measuredangle P_2OP_3$ and $\measuredangle P_2P_1P_3$ are supplementary. But also $\measuredangle P_2AP_3$ and $\measuredangle P_2OP_3$ are supplementary (A is any point on α_1); hence $\measuredangle P_2P_1P_3 = \measuredangle P_2AP_3$. From this follows immediately that the circle α_4 through $P_1P_2P_3$ is equal to the circle α_1, and consequently to α_2 and α_3. As the sum of the six angles in the equality

$$\measuredangle OP_2P_1 + \measuredangle OP_2P_3 + \measuredangle OP_3P_2 = \measuredangle OP_3P_1 + \measuredangle OP_1P_3 + \measuredangle OP_1P_2$$

is equal to the sum of the angles in the triangle $P_1P_2P_3$, the left and right hand sides are equal to a right angle, *i. e.*, P_2P_3Q is a right triangle, and consequently P_3Q is perpendicular to P_1P_2. Likewise P_2O and P_1O prolonged are perpendiculars to P_1P_3 and P_3P_2, respectively; and O is the orthocenter of the triangle $P_1P_2P_3$. In a similar manner it can be shown, that P_1, P_2, P_3, are orthocenters of the corresponding triangles OP_2P_3, OP_3P_1, OP_1P_2.

[1] For figures in which O is without the triangle $P_1P_2P_3$, the proposition can be proved in a similar manner by angular relations.